# Foundational Reasoning Abilities that Promote Coherence in Students' Function Understanding

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The concept of function is central to undergraduate mathematics, foundational to modern mathematics, and essential in related areas of the sciences. A strong understanding of the function concept is also essential for any student hoping to understand calculus – a critical course for the development of future scientists, engineers, and mathematicians.

Since 1888, there have been repeated calls for school curricula to place greater emphasis on functions (College Entrance Examination Board, 1959; Hamley, 1934; Hedrick, 1922; Klein, 1883; National Council of Teachers of Mathematics, 1934, 1989, 2000). Despite these and other calls, students continue to emerge from high school and freshman college courses with a weak understanding of this important concept (Carlson, 1998; Carlson, Jacobs, Coe, Larsen & Hsu, 2002; Cooney & Wilson, 1996; Monk, 1992; Monk & Nemirovsky, 1994; Thompson, 1994a). This impoverished understanding of a central concept of secondary and undergraduate mathematics likely results in many students discontinuing their study of mathematics. The primarily procedural orientation to using functions to solve specific problems is absent of meaning and coherence for students and has been observed to cause frustration in students (Carlson, 1998). We advocate that instructional shifts that promote rich conceptions and powerful reasoning abilities may generate students' curiosity and interest in mathematics, and subsequently lead to increases in the number of students who continue their study of mathematics.

This article provides an overview of essential processes involved in knowing and learning the function concept. We have included discussions of the reasoning abilities involved in understanding and using functions, including the dynamic conceptualizations needed for understanding major concepts of calculus, parametric functions, functions of several variables, and differential equations. Our discussion also provides information about common conceptual obstacles to knowing and learning the function concept that students have been observed encountering. We make frequent use of examples to illustrate the 'ways of thinking' and major understandings that research suggests are essential for students' effective use of functions during problem solving, and that are needed for students' continued mathematics learning. We also provide some suggestions for promising approaches for developing a deep and coherent view of the concept of function.

## Why Is The Function Concept So Important?

Studies have revealed that learning the function concept is complex, with many high performing undergraduates (e.g., students receiving course grades of A in calculus) possessing weak function understandings (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Carlson, 1998; Thompson, 1994a). We are beginning to understand that the conceptions and reasoning patterns needed for a strong and flexible understanding of functions are more complex than is typically assumed by designers of curriculum and instruction (Breidenbach et al., 1992; Carlson, 1998; Thompson, 1994a). Students who

think about functions only in terms of symbolic manipulations and procedural techniques are unable to comprehend a more general mapping of a set of input values to a set of output values; they also lack the conceptual structures for modeling function relationships in which the function value (output variable) changes continuously in tandem with continuous changes in the input variable (Carlson, 1998; Monk & Nemirovsky, 1994; Thompson, 1994a). These reasoning abilities have been shown to be essential for representing and interpreting the changing nature of a wide array of function situations (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Thompson, 1994a); they are also foundational for understanding major concepts in advanced mathematics (Carlson, Smith & Persson, 2003; Cottrill et al., 1996; Kaput, 1992; Rasmussen, 2000; Thompson, 1994a; Zandieh, 2000).

It is noteworthy that many of the reform calculus texts of the early 90's, e.g., Ostabee & Zorn (1997), Harvard Calculus (Hughes-Hallet & Gleason, 1994), and C4L (Dubinsky, Schwingendorf, & Mathews, 1994), included a stronger conceptual orientation to learning functions. Such past curriculum development projects and the educational research literature are pointing the way for future curricular interventions to assist students in developing a robust function conception – a conception that begins with a view of function as an entity that accepts input and produces output, and progresses to a conception that enables reasoning about dynamic mathematical content and scientific contexts. Research suggests that the predominant approach to calculus instruction is not achieving the foundational understandings and problem solving behaviors that are needed for students' continued mathematical development and course taking (Carlson, 1998, 1999, 2003; Oehrtman, 2002). It is our view that the mathematics community is ready for a careful rethinking of the precaclulus and calculus curriculum – one that is driven by past work of mathematicians, as well as the broad body of research on knowing and learning function and major concepts of calculus. It is also our view that if algebraic and procedural methods were more connected to conceptual learning, students would be better equipped to apply their algebraic techniques appropriately in solving novel problems and tasks.

*Why is the Function Concept So Difficult for Students to Understand?*

As students move through their school and undergraduate mathematics curricula, they are frequently asked to manipulate algebraic equations and compute answers to specific types of questions. This strong emphasis on procedures without accompanying activities to develop deep understanding of the concept has not been effective for building foundational function conceptions – ones that allow for meaningful interpretation and use of function in various representational and novel settings. Even understanding functions in terms of input and output can be a major challenge for many students. As one example, 43% of A-students at the completion of college algebra attempted to find $f(x+a)$ by adding $a$ onto the end of the expression for $f$ rather than substituting $x+a$ into the function (Carlson, 1998). When probed to explain their thinking, they typically provided some memorized rule or procedure to support their answers. Clearly these students were not thinking of $x+a$ as a value of the function's argument at which the function is being evaluated. Another misconception is thinking that constant functions (e.g., $y=5$) are not functions because they do not vary. Not viewing $y = 5$ as an example of a function can become problematic for students; as one

example, when considering equilibrium solution functions for differential equations such as d$y$/d$t$ = 2$y$($y$-5) (Rasmussen, 2000). In one study, only 7% of A-students in a college algebra course could produce a correct example of a function all of whose output values are equal to each other, while 25% of A-students in second semester calculus produced $y = x$ as an example (Carlson, 1998). Even more problematic, students often view functions simply as two expressions separated by an equal sign (Thompson, 1994b). Such an impoverished understanding of functions is insufficient to serve as a base for a rich understanding of more advanced mathematics.

It is also common for developing students to have difficulty distinguishing between an algebraically defined function and an equation (Carlson, 1998). This is not surprising if one considers the various uses of the equal sign and the fact that many instructors refer to a formula as an equation. For the student, this ambiguous use of the word equation appears to cause difficulty for them in distinguishing between the use of the equal sign as a means of defining a relationship between two varying quantities and a statement of equality of two expressions. Our recent work has shown that students benefit from an explicit effort to help them distinguish between functions and equations. The first two authors have developed instructional interventions that promote students' thinking about an equation as a means of equating the output values of two functions, and the act of solving an equation as finding the input value(s) where the output values of these functions are equal.

Many students also tend to believe that all functions should be definable by a single algebraic formula. This focus often hinders flexible thinking about function situations and can lead to erroneous conclusions such as thinking that all functions must always behave "nicely" in some sense (Breidenbach et al., 1992). For example, many students tend to argue that a piecewise defined function like $f(x) = \begin{cases} 0, & x \le 0; \\ e^{-1/x^2}, & x > 0, \end{cases}$ is actually two separate functions or that a function such as Dirichlet's example, $g(x) = \begin{cases} 1, & x \text{ is rational}, \\ 0, & x \text{ is irrational}, \end{cases}$ is not even a function at all because it "behaves badly." Similarly, many students have difficulty conceiving of different formulas representing the same function, as in the examples $f_1(n) = n^2$ and $f_2(n) = \sum_{k=1}^{n}\left[2k-1\right]$, which define the same function on the natural numbers, albeit through very different algebraic operations. Many students also tend to assume that functions are linear or quadratic in cases where this assumption is unwarranted, expecting for example, that any "u-shaped" graph is a parabola (Schwarz & Hershkowitz, 1999). These tendencies are perhaps not so surprising when we consider that functions are typically introduced in the school curriculum through specific function types. As such, a working definition in which functions are equated with formulas is perfectly reasonable, and even mirrors the historical understanding of mathematicians like Euler, Bernoulli, Lagrange, and d'Alembert (Kleiner, 1989; Sierpinska, 1992). It is not, however, the view that the Euler himself, and subsequently the mathematics community in general, ultimately found to be most useful. The modern definition of function was motivated largely by debates between d'Alembert and Euler on the nature of a solution to the vibrating string differential equation (Luzin,

1998a, 1998b) and by Cauchy's and others' attempts to decide the conditions under which a limit of a sequence of continuous functions is a continuous function (Boyer, 1968; Lakatos, 1976). Thus, to use the modern definition of function in an introduction to the function concept is to present students with a solution to problems of which they cannot conceive. We recommend that school curricula and instruction include a greater focus on understanding ideas of covariation and multiple representations of covariation (e.g., using different coordinate systems), and that more opportunities be provided for students to experience diverse function types emphasizing multiple representations of the same functions. College curricula could then build on this foundation. This would promote a more flexible and robust view of functions – one that does not lead to inadvertently equating functions and formulas.

Another common difficulty for students is distinguishing between visual attributes of a physical situation and similar perceptual attributes of the graph of a function that models the situation. When dealing with functions as models of concrete situations, there are often topographical structures within the real-world setting itself (e.g., the curves of a racetrack, the elevation of a road traveling across hilly terrain, or the shape of a container being filled with liquid) that students see as being reflected in the function's graph. The considerable salience of these physical features often creates confusion, even for students with a strong understanding of function. Several types of errors can be traced to conflating the shape of a graph with visual attributes of the situation (Carlson, 1998; Monk, 1992; Monk & Nemirovsky, 1994). Consider the following problem:

> The following diagram is the side-view of a person cycling up and over a hill. Draw a graph of speed vs. position along the path.

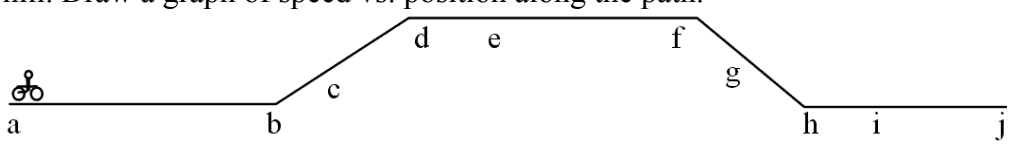



**Figure 1.** A problem in which students must distinguish between visual features of a situation and representational features of a graph. (From Monk, 1992).

In response to this problem, many students tend to copy features directly from the diagram into their graph (Monk, 1992). Correctly interpreting the situation is not a conceptually trivial task. A student must ignore the fact that the picture looks like a graph, think of how riding uphill (for example) affects the speed of the cyclist, then, while ignoring the shape of the hill in the picture, determine how to represent the result graphically.

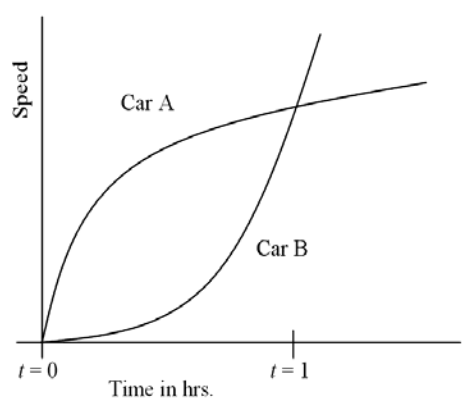



**Figure 2.** Students fail to interpret the function information conveyed by the graph.

When interpreting graphs such as the one in Figure 2, students often confuse velocity for position (Monk, 1992) since the curves are laid out spatially, and position refers to a spatial property. This confusion leads to erroneous

claims such as: the two cars collide at $t = 1$ hour or that Car B is catching up to Car A between $t = .75$ hour and $t = 1$ hours. In one study, 88% of students who had earned an A in college algebra made such mistakes, as did 63% of students earning an A in second semester calculus, and 42% of students earning an A in their first graduate mathematics course (Carlson, 1998).

In both these examples, students are thinking of the graph of a function as a picture of a physical situation rather than as a mapping from a set of input values to a set of output values. Developing an understanding of function in such real-world situations that model dynamic change is an important bridge for success in advanced mathematics.

Students' weak understandings of functions have also been observed in their inability to express function relationships using function notation. When asked to express *s* as a function of *t*, many high performing precalculus students did not know that their objective was to write a formula in the form of "*s* = <some expression containing a *t*>." Some students have also exhibited weaknesses in knowing what each symbol in an algebraically defined function means. Even in the case of a simple function such as $f(x) = 3x$, many students are unaware that the parentheses serves as a marker for the input, that $f(x)$ represents the output values, that *f* is the name of the function, and that $3x$ specifies how the input $x$ is mapped to the output $f(x)$. Such weak understandings and highly procedural orientations appear to contribute to students' inability to move fluidly between various function representations, such as the inability to construct a formula given a function situation described in words (Carlson, 1998).

*Dynamic Conceptualizations Needed for Understanding and Using Functions*

In our work to develop and validate the *Precalculus Concept Assessment Instrument*[1] (Carlson, Oehrtman, & Engelke, submitted), the first two authors found that students' ability to respond correctly to a diverse set of function-focused tasks is tightly linked to two types of dynamic reasoning abilities. First, as mentioned above, students must develop an understanding of functions as general processes that accept input and produce output. Second, they must be able to attend to both the changing value of the output and rate of its change as the independent variable is varied through an interval in the domain.

Understanding limits and continuity requires one to make judgments about the behavior of a function over intervals of arbitrarily small sizes. Conceptualizations based on "holes," "poles," and "jumps" as gestalt topographical features (corresponding to removable discontinuity, vertical asymptotes, and jump or one-sided discontinuity, respectively) can lead to misconceptions in more complex limiting situations, such as the definitions of the derivative and definite integral. For example, students can develop an intuitive understanding of the Fundamental Theorem of Calculus with which they explain why the derivative of the volume of a sphere ($v = \frac{4}{3}\pi r^3$) with respect to the length of its radius is its surface area. However, most of these students cannot explain why the same is

[1] The Precalculus Concept Assessment Instrument is a 25-item multiple choice instrument for assessing students' understanding of the major aspects of the function concept that are foundational for success in beginning calculus. The answer choices include the correct answer and the common misconceptions that have been expressed by students in research studies (e.g., interviews that have probed students' thinking when providing specific responses to conceptually based tasks).

not true for the volume of a cube ( $v = s^3$ ) with respect to the length of its side (Oehrtman, 2002). In order to resolve such results conceptually, one must be able to coordinate images of changes in the "radius" with the corresponding changes in the volume over a range of small variations. For such variations, students must then be able to imagine the computation of rate of change of volume and see its connection to the computation of surface area.

To understand the relationship between average and instantaneous rates and the graphical analog between secant and tangent lines, a student must first conceive of an image as in Figure 3a (Monk, 1987). By employing *covariational reasoning* (e.g., coordinating an image of two varying quantities and attending to how they change in relation to each other), the student is able to transform the image and reason about values of various parameters as the configuration changes. Being able to answer questions that require covarying two quantities, i.e., "When point *Q* moves toward *P*, does the slope of *S* increase or decrease?", is significantly more difficult than being able to answer questions about the value of a function at a single point.

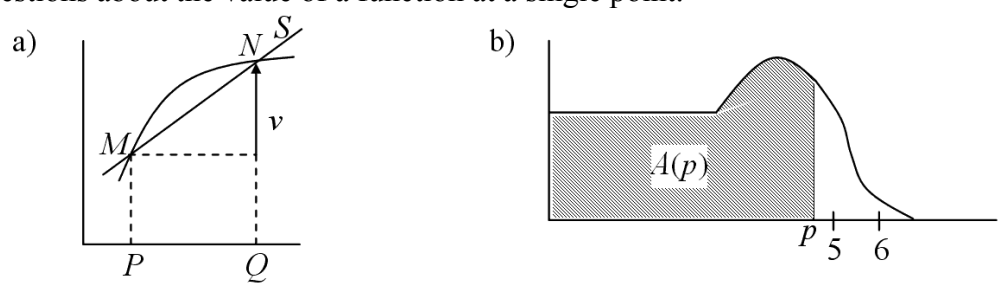



**Figure 3.** Foundational images for the definitions of a) the derivative and b) the definite integral.

Analyzing the changing nature of an instantaneous rate also requires the ability to conceive of functional situations dynamically. Consider the following question based on a classic related rates problem in calculus:

> From a vertical position against a wall, the bottom of a ladder is pulled away at a constant rate. Describe the speed of the top of the ladder as it slides down the wall.

Reasoning about this situation conceptually is difficult for calculus students even when they are given a physical model and scaffolding questions (Monk, 1992) and is similarly challenging for beginning graduate students in mathematics (Carlson, 1999). The standard calculus curriculum presents accumulation in terms of methods of determining static quantities -- such as the area of an irregular region of the plane, or the distance traveled over a fixed amount of time given a changing velocity. Students imagine themselves approximating an area of a region. The area happens to be defined by a graph, but the task, to them, is essentially the same as approximating the area of a circle with triangles emanating from the circle's center. Equally important, however, is a dynamic view in which an accumulated total is changing through continual accruals (Kaput, 1994; Thompson, 1994a). For example, in a typical "area so far" function as in Figure 3b, this involves being able to mentally imagine the point *p* moving to the right by

adding slices of area at a rate proportional to the height of the graph. This requires students to engage in covariational reasoning (Carlson, Smith & Persson, 2003) and is significantly more difficult for students than evaluating and even comparing areas at given points (Monk, 1987), for instead of asking them to conceptualize $m = \int_a^b f(x)\,dx$, we are asking them to conceptualize $F(x) = \int_a^x f(t)\,dt$.

In interviews with over 40 precalculus level students, the first two authors found that students who consistently verbalized a view of function as an entity that accepts input and produces output were able to reason effectively through a variety of function-related tasks. For example, these students, when asked to find $f(g(x))$ for specific values of *x*, given in either a table or words that defined the functions *f* and *g*, described a process of inputting a value into *g*, with the output of *g* becoming the input of *f,* and this output providing an output for the composite function $f \circ g$. However, students who provided an incorrect answer to this question were typically attempting to employ some memorized procedure. Without understanding, they invariably made a crucial mistake along the way such as interpreting $f(g(3))$ as meaning "the value of *f* when *g* is three," and by mistaking the *output* of *g* to be 3, arriving at $f(3)$ as an answer. As another example, when asked to solve the equation, $f(x) = 7$, given the graph of *f,* students who viewed this problem as a request to reverse the function process to determine the input associated with an output of *f*, had no difficulty responding to this task. Surprisingly, only 38% of 1196 students (550 college algebra and 646 precalculus) provided a correct answer at the completion of their courses. Those unable to provide a correct answer appeared to be applying memorized procedures – they did not speak about a function as a more general mapping of a set of input values to a set of output values. Their impoverished function view was also revealed by their inability to explain the meaning of function composition and function inverse in other settings and their inability to apply function composition to define an algebraic formula for a function situation (e.g., to define area as a function of time for a circle whose radius is expanding at 7 cm per second).

According to several studies, calculus students are slow to develop an ability to interpret varying *rates of change* over intervals of a function's domain. (Carlson, 1998; Kaput, 1992; Monk, 1992; Monk & Nemirovsky, 1994; Nemirovsky, 1996; Tall, 1992; Thompson, 1994a). According to Thompson (1994a), once students are adept at imagining expressions being evaluated continually as they "run rapidly" over a continuum, the groundwork has been laid for them to reflect on a set of possible inputs in relation to the set of corresponding outputs (p. 27). Such a *covariation view* of function has also been found to be essential for understanding central concepts of calculus (Cottrill et al., 1996; Kaput, 1992; Thompson, 1994b; Zandieh, 2000) and for reasoning about average and instantaneous rates of change, concavity, inflection points, and their real-world interpretations (Carlson, 1998; Monk, 1992).

The following section provides additional elaboration of these essential process and covariational understandings of functions.

*The Action and Process Views of Functions – A More Formal Examination*

Developmental research has provided insights about the reasoning patterns essential for success in collegiate mathematics. As we have previously discussed, investigations of students' function knowledge have consistently revealed that students' underlying conceptual view is important. Researchers have formalized these consistent observations by introducing terms for referencing specific types of conceptual views and their development. Specifically, students must move from what is called an *action view of functions* to what is called a *process view of functions*.

According to Dubinsky & Harel (1992),

> An *action* conception of function would involve the ability to plug numbers into an algebraic expression and calculate. It is a static conception in that the subject will tend to think about it one step at a time (e.g., one evaluation of an expression). A student whose function conception is limited to actions might be able to form the composition of two functions, defined by algebraic expressions, by replacing each occurrence of the variable in one expression by the other expression and then simplifying; however, the students would probably be unable to compose two functions that are defined by tables or graphs. (pp. 85)

Students whose understanding is limited to an action view of function experience several difficulties. For example, an inability to interpret functions more broadly than by the computations involved in a specific formula results in misconceptions such as believing that a piecewise function is actually several distinct functions, or that different algorithms must produce different functions. More importantly, reasoning dynamically is difficult because it requires one to be able to disregard specific computations and to be able to imagine running through all input-output pairs simultaneously. This ability is not possible with an action view in which each individual computation must be explicitly performed or imagined. Furthermore, from an action view, input and output are not conceived except as a result of values considered one at a time, so the student cannot reason about a function acting on entire intervals. Thus, not only is the complex reasoning required for calculus out of reach for these students, but even simple tasks like conceiving of domain and range as entire sets of inputs and outputs is difficult.

Without a generalized view of inputs and outputs, students cannot think of a function as a process that may be reversed (to obtain the inverse of a function) but are limited to understanding only the related procedural tasks such as switching $x$ and $y$ and solving for $y$ or reflecting the graph of $f$ across the line $y = x$ (Figures 4a and 4b). This procedural approach to determining "an answer" has little or no real meaning for the student unless he or she also possesses an understanding as to why the procedure works. Students with an action view often think of a function's graph as being only a curve (or fixed object) in the plane; they do not view the graph as defining a general mapping of a set of input values to a set of output values. As such, the location of points, the vertical line test, and the "up and over" evaluation of functions on a graph are concepts only about the geometry of the graph, not about the more general mapping that is conveyed by the function, or the meaning that is conveyed by inverting the process for a function that represents a real-world situation. Similarly, with an action view, composition is generally seen simply as an algebra problem in which the task is to substitute one expression for every instance of $x$ into some other expression. An understanding of why these

procedures work or how they are related to composing or reversing functions is generally absent.

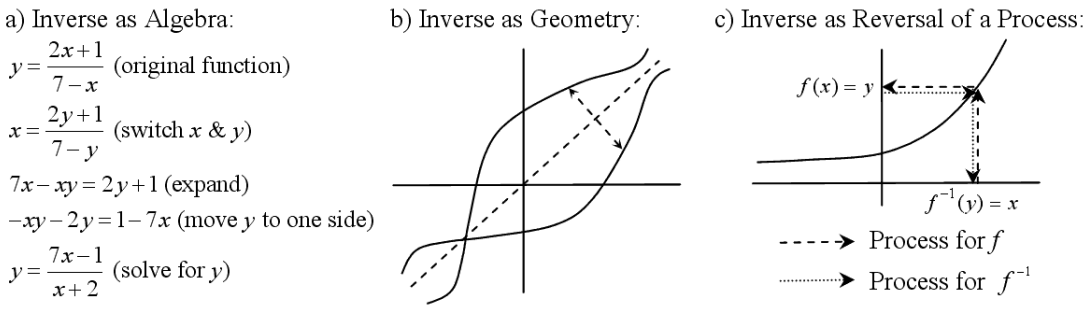



**Figure 4.** Various conceptions of the inverse of a function as a) an algebra problem, b) a geometry problem, and c) the reversal of a process. The first two of these are common among students but, in isolation, do not facilitate flexible and powerful reasoning about functional situations.

Students who possess only the procedural orientations of Figures 4a and b, without understanding why the procedures work, are not likely to recognize even simple situations in which these procedures should be applied. Curriculum and instruction has not been broadly effective in building these connections in students' understanding. A recent study of over 2000 precalculus students at the end of the semester (Carlson et al., submitted) showed that only 17% of these students correctly determined the inverse of a function for a specific value, given a table of function values.

In contrast to the conceptual limitations of an action view, Dubinsky and Harel (1992) state:

> A *process* conception of function involves a dynamic transformation of quantities according to some repeatable means that, given the same original quantity, will always produce the same transformed quantity. The subject is able to think about the transformation as a complete activity beginning with objects of some kind, doing something to these objects, and obtaining new objects as a result of what was done. When the subject has a process conception, he or she will be able, for example, to combine it with other processes, or even reverse it. Notions such as 1-1 or onto become more accessible as the student's process conception strengthens. (p. 85)

With such a process view, students are freed from having to imagine each individual operation for an algebraically defined function. For example, given the function on the real numbers defined by $f(x) = x^2 + 1$, the student can imagine a set of input values that are mapped to a set of output values by the defining expression for *f*. In contrast, students with an action view see the defining formula as a procedure for finding an answer for a specific value of *x*; they view the formula as a set of directions: square the value for *x* then add one to get the answer. A student with a process view can conceive of the entire process as happening to all values at once, and is able to conceptually run through a continuum of input values while attending to the resulting impact on output values. This

is precisely the ability required for covariational reasoning introduced above and discussed more fully in the following section. In Table 1 we provide a characterization of "action views" of functions and their corresponding "process views."

**Table 1**
*Action and Process Views of Functions*

| Action View | Process View |
|---|---|
| A function is tied to a specific rule, formula, or computation and requires the completion of specific computations and/or steps. | A function is a generalized input-output process that defines a mapping of a set of input values to a set of output values. |
| A student must perform or imagine *each action.* | A student can imagine the *entire process* without having to perform each action. |
| The "answer" depends on the formula. | The process is independent of the formula. |
| A student can only imagine a single value at a time as input or output (e.g., $x$ stands for a specific number). | A student can imagine all input at once or "run through" a continuum of inputs. A function is a transformation of entire spaces. |
| Composition is substituting a formula or expression for $x$. | Composition is a *coordination* of two input-output processes; input is processed by one function and its output is processed by a second function. |
| Inverse is about algebra (switch $y$ and $x$ then solve) or geometry (reflect across $y=x$). | Inverse is the *reversal of a process* that defines a mapping from a set of output values to a set of input values. |
| Finding domain and range is conceived at most as an algebra problem (e.g., the denominator cannot be zero, and the radicand cannot be negative). | Domain and range are produced by operating and reflecting on the set of all possible inputs and outputs. |
| Functions are conceived as static. | Functions are conceived as dynamic. |
| A function's graph is a geometric figure | A function's graph defines a specific mapping of a set of input values to a set of output values. |

Understanding even the basic idea of equality of two functions requires a generalization of the input-output process, (i.e., the ability to imagine the pairing of inputs to unique outputs without having to perform or even consider the means by which this is done). Students may then come to understand that any means of defining the same relation is the same function. That is, a function is not tied to specific computations or rules that define how to determine the output from a given input. For example, the rules

$n \overset{f}{\mapsto} n^2$ vs. $n \overset{g}{\mapsto} \sum_{k=1}^{n} [2k-1]$ look different; yet produce the same results (and thus define the same function) on the natural numbers.

Students with a process view are also better able to understand aspects of functions such as composition and inverses. They are consistently able to correctly answer conceptual and computational questions about composition in a variety of representations by coordinating output of one process as the input for a second process. Similarly, students conceiving of an inverse as reversing the function process so that the old outputs become the new inputs and vice-versa (Figure 4c), or by asking "What does one have to do to get back to the original values?" were able to correctly answer a wide variety of questions about inverse functions (Carlson et al., submitted).

A process view of function is crucial to understanding the main conceptual strands of calculus (Breidenbach et al., 1992; Monk, 1987; Thompson, 1994a). For example, the ability to coordinate function inputs and outputs dynamically is an essential reasoning ability for limits, derivatives and definite integrals. In order to understand the definition of a limit, a student must coordinate an entire interval of output values, imagine reversing the function process, and determine the corresponding region of input values. The action of a function on these values must be considered simultaneously since another process (one of reducing the size of the neighborhood in the range) must be applied while coordinating the results. Unfortunately, most pre-calculus students do not develop beyond an action view, and even strong calculus students have a poorly developed process view that often leads only to computational proficiency (Carlson, 1998). With intentional instruction, however, students can develop a more robust process view of function (Carlson et al., submitted; Dubinsky, 1991; Sfard, 1991).

Certainly not every aspect of an action view of functions is detrimental to students' understanding, just as the acquisition of a process view does not ensure success with all functional reasoning. However, a process view of functions is crucial for developing rich conceptual understandings of the content in an introductory calculus course. The promotion of the more general 'ways of thinking' that we have advocated should result in producing curricula that are more effective for promoting conceptual structures for students' continued mathematical development.

*Fostering a Process View of Functions*

We offer the following general recommendations for promoting students' development of a process view of functions:

*Ask students to explain basic function facts in terms of input and output* For example, ask students to determine whether $(f \circ g)^{-1}$ is $f^{-1} \circ g^{-1}$ or $g^{-1} \circ f^{-1}$ and explain their reasoning. In the process, most will initially struggle to decide which of the diagrams in Figure 5 represents $f \circ g$. Determining both the correct diagram and the correct formula for the inverse encourages students to think in terms of a general input-output process. As another example, students typically

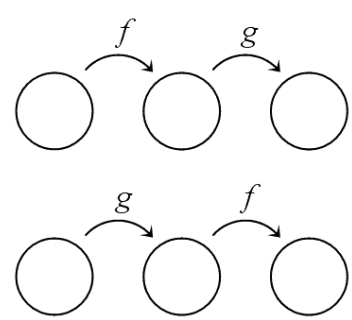



**Figure 5.** Which diagram represents $f \circ g$? What is its inverse?

learn to carry out rote procedures when asked to solve equations such as $f(x) = 6$ for some specified function f; but asking them to find the input value(s) for which the function's output is 6 (both algebraically and graphically) promotes an understanding that solving an equation can be seen as the reversal of a function process. As yet another example, students typically memorize (without understanding) that the graph of a function g given by $g(x) = f(x + a)$ is shifted to the left of the graph of *f*, but asking them to discover or interpret this statement as meaning "the output of g at every *x* is the same as the output of f at every $x + a$" will give them a more powerful way to understand this idea and reinforce a process view of functions. Ask students to determine the domain and range of functions based on the problem context, and relate this to answers (possibly different) derived from algebraic constraints alone. Other possibilities include asking students to explain why composition is associative, to develop the definition of a periodic function on their own, or to graph and explain the results of simple function arithmetic.

*Ask about the behavior of functions on entire intervals in addition to single points.* Focusing on the image of a function applied to an infinite set also encourages students to think in terms of a general process. Students should be asked to coordinate such judgments with basic compositions and inverses, asking, for example, for the length of an interval after being transformed by two linear functions. Similarly, ask students to find preimages of intervals as in the definition of limit or continuity and to reverse the process of a function even if it is not invertible (e.g., find the preimage of 1 under $f(x) = x^2$).

*Ask students to make and compare judgments about functions across multiple representations.* Such questions should include multiple algebraic representations to reinforce the independence from a formula as well as the standard representations of graphs, tables, and verbal descriptions. Students should make such determinations; then compare the results for consistency, justifying or discovering why they are the same. For example, asking how the various techniques of inverting a function are related reinforces seeing a reflection across the line $y = x$ as switching the roles of independent and dependent variable, of input and output. Also helpful are predictions about how a graph will look based on how a real-world quantity is changing across its domain, requiring simultaneous attention to multiple input-output pairs and translation between representations. As an example, when asking students to solve standard problems such as 'define the area as a function of time for a circle whose radius is expanding at 7 cm per second,' ask students to begin by constructing a dynamic image of the situation via a computer program or by drawing a picture; then ask them to label using algebraic symbols the varying quantities in the situation. After recognizing the 7cm change in the radius per second can be represented algebraically by labeling the varying length of the radius with the formula $r = 7s$ on the picture, prompt them to determine how to relate area to time in seconds. You could also ask your students to graph the resulting function, A = $\pi(7s)^2$, and determine the average rate of change of the circle as *s* changes from 3 to 4; then as *s* changes from 4 to 5; then as *s* changes from 5 to 6; then as *s* changes from 6 to 7; then ask them to explain in the context of the growing circle what these average rates imply about how the area of the circle is growing over the time interval from $s$=3 to $s$ = 7 (For additional discussion of the complexities involved in acquiring a flexible view of

variable; as an unknown, varying quantity, placeholder, etc., see the Jacobs and Trigerous chapter in this volume).

*Building on the Process View of Function: Applying Covariational Reasoning*

As students begin to explore dynamic function relationships such as how speed varies with time or how the height of water in a bottle varies with volume, they will need to begin considering *how* one variable (often the dependent variable) changes while imagining changes in the other (the independent variable). When coordinating such changes, one must be able to represent and interpret important features in the shape of a graph of a dynamic function event. As a very simple example, a student who has a strong process view of function, might see the algebraic formula, $A(s) = s^2$ as a means of determining the area of a square for a set of possible input values. She would be viewing the function as an entity that accepts any side length $s$ as input, to produce an output value for the area $A$. She would have no difficulty determining the side of the square for given values of the area (reversing the process) or with using any particular representation of this function situation (algebraic, tabular, graphical). In this context, the student may begin to notice that as the value of $s$ increases, the value of $A$ increases. By exploring numerical patterns and/or constructing a graph of this function, the student may also observe that as one steps through positive integer values for $s$, the amount of increase of $A$ is getting larger and larger. He or she may also notice that as $s$ increases continuously, the area is growing faster and faster. By constructing a graph to represent this function relationship, the student may observe that, when $s$ is greater than 0, the slope of the graph gets steeper as $s$ increases. When asked to explain why the graph gets steeper, the student would also be able to unpack the notion of slope (steepness) by describing the relative change of the input (side) and output (area), while stepping through values of $s$.

## The Covariation Framework

Our work to characterize the thinking involved in reasoning flexibly about dynamically changing events has led to our decomposing *covariational reasoning* into five distinct mental actions (Carlson et al., 2002). This decomposition has been useful for guiding the development of curricular modules to promote covariational reasoning in students. These five categories of mental actions (Table 2) describe the reasoning abilities involved in meaningful representation and interpretation of a graphical model of a dynamic function situation. In our work, the first two authors have developed beginning calculus modules that include tasks and prompts to promote these ways of thinking in students. After three iterations of refining these modules (based on our analysis of data of students' reasoning when working through these modules), we are observing dramatic gains in beginning calculus students' *covariational reasoning* abilities over the course of one semester.

The initial image described in the framework for covariational reasoning is one of two variables changing simultaneously. This loose association undergoes multiple refinements as the student moves toward an image of increasing and decreasing rate over the entire domain of the function (Table 2).

**Table 2**
Mental Actions of the Covariation Framework

| ***Mental Action*** | ***Description of Mental Action*** | ***Behaviors*** |
|---|---|---|
| *Mental Action 1 (MA1)* | Coordinating the dependence of one variable on another variable | • Labeling the axes with verbal indications of coordinating the two variables (e.g., y changes with changes in x) |
| *Mental Action 2 (MA2)* | Coordinating the direction of change of one variable with changes in the other variable | • Constructing a monotonic straight line<br>• Verbalizing an awareness of the direction of change of the output while considering changes in the input |
| *Mental Action 3 (MA3)* | Coordinating the amount of change of one variable with changes in the other variable | • Plotting points/constructing secant lines<br>• Verbalizing an awareness of the amount of change of the output while considering changes in the input |
| *Mental Action 4 (MA4)* | Coordinating the average rate-of-change of the function with uniform increments of change in the input variable | • Constructing secant lines for contiguous intervals in the domain<br>• Verbalizing an awareness of the rate of change of the output (with respect to the input) while considering uniform increments of the input |
| *Mental Action 5 (MA5)* | Coordinating the instantaneous rate-of-change of the function with continuous changes in the independent variable for the entire domain of the function | • Constructing a smooth curve with clear indications of concavity changes<br>• Verbalizing an awareness of the instantaneous changes in the rate-of-change for the entire domain of the function (direction of concavities and inflection points are correct) |

In our work to study and promote students' emerging covariational reasoning abilities, we have found that the ability to move flexibly between mental actions 3, 4 and 5 is not trivial for students. We have also observed that many precalculus level students only employ Mental Action 1 and Mental Action 2 when asked to construct the graph of a dynamic function situation.

When prompting students to construct the graph of the height as a function of the amount of water in a bottle (Figure 6), the first two authors found that many precalculus students appropriately labeled the axes (MA1) and then constructed an increasing straight line (MA2). When prompted to explain their reasoning, they frequently indicated that "as more water is put into the bottle, the height of the water rises (MA2)." These students were clearly not attending to the amount of change of the height of the water level or the rate at which the water was rising.

Imagine this bottle filling with water. Sketch a graph of the height as a function of the amount of water that's in the bottle.

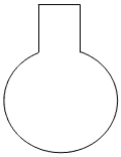


**Figure 6.** The bottle problem.

We have observed that calculus students frequently provided a strictly concave up graph in response to this question (Carlson, 1998; Carlson et al., 2002). When probed to explain their reasoning, a common type of justification was, "as the water is poured in it gets higher and higher on the bottle (MA2)." In contrast, other students who were starting to be able to construct an appropriate graph began coordinating the magnitude of changes in the height with changes in the volume (MA3). This is exemplified in the strategy of imagining pouring in one cup of water at a time and coordinating the resulting change in height based on how "spread out" that layer of water is.

Other students have demonstrated the ability to speak about the average rate of change locally for a specific interval of a function's domain (MA4) but were unable to explain how the rate changes over the domain of the function. Even when calculus students produced a graph that was correct, they commonly had difficulty explaining what was conveyed by the inflection point and why the graph was "smooth" (in particular, why it is $C^1$ rather than piecewise linear). Students frequently exhibited behaviors that gave the appearance of engaging in Mental Action 5 (e.g., construction of a smooth curve with the correct shape), however when prompted to explain their reasoning, they expressed that they had relied on memorized facts to guide their constructions. They were relying on apparent facts such as "faster means steeper" and "slower means less steep," but they were unable to explain why this was true.

*Engaging Covariational Reasoning Through Analysis Of Function Situations*

We offer the following suggestions for strengthening students' covariational reasoning abilities:

*Generally, ask questions associated with each of the mental actions.* For orientation to any problem, MA1 skills and basic function awareness can be addressed by asking what values are changing and what variable(s) influence the quantity of interest (i.e., the dependent variable). Is there a single variable that determines that quantity's values? How are the variables related and in what representations can this relationship be expressed? For MA2, ask whether a function increases or decreases if the independent variable is increased (or decreased). Expect students to make such judgments from multiple representations. At an MA3 level, ask students to make judgments about amounts of change in the function for constant increments of the independent variable. For a dynamic situation, have students draw diagrams representing changes from one output variable to the other for each of two nearby intervals of the input variable, and represent these changes pictorially and algebraically. Ask students to interpret these representations in terms of rate of change in the problem context. To foster MA4 thinking, have students compute several average rates using various representations and find various interpretations for these values and explicitly discuss the meaning of units such as meters per second and even non-temporal rates such as square inches per inch or degrees Kelvin per meter. For MA5, ask students to anticipate second derivative information based on the problem context, e.g., whether the force of gravity between two celestial objects will increase at an increasing rate or at a decreasing rate with respect to a decreasing distance between them. Ask students to describe the rate of change of a function event as the independent variable continuously and dynamically varies through the domain. Ask where inflection points are, what events they correspond to in real-world situations, and how these points are interpreted in terms of changing rate of change.

*Ask for clarification of rate of change information in various contexts and representations.* Expect students to explain statements about rates in real-world contexts from algebraic or graphical information, e.g., why does a steeper graph mean the quantity represented by the function is increasing faster? Push beyond students' initial, simplified statements such as "the rate of change of position" that ignore the role of time. Require explication of *both* variables involved and relationships about changes in both quantities. Finally, a student may be able to make statements indicative of a Mental Action 5 by attending only to the geometry of the curve and associated phrases such as "increasing at a decreasing rate." Ask them to unpack such statements in terms of the underlying mental actions, in this case perhaps prompts that reveal if they understand what they mean by the phrase "increasing at a decreasing rate." Unpacking what may be pseudo conceptual knowledge—knowledge that has been memorized and is not based on an underlying conceptual structure and understanding, can be achieved by posing pointed questions that prompt students to reveal their underlying conceptions (e.g., why is the graph concave up or why is the curve "smooth" rather than piecewise linear?) Such questions typically reveal if the student is merely spouting a memorized rule or fact, or if the statement is supported by an understanding of why the rule or statement is true.

*Extending Ideas of Covariation To Higher Dimensions*

The idea of covariation is fundamentally that of parametric functions. As one imagines scanning through values of one variable and keeping track of values of another variable, one is essentially imagining the parametric function $(x,y) = (t,f(t))$. Once students have developed the ability to reason covariationally, it is a natural (but not small) step to reason about functions defined parametrically by $(x,y) = (f(t), g(t))$. For example, the graph in Figure 7 is $(x,y) = (\sin 10t, \cos 20t)$, $0 \le t \le 1$. Students can conceptualize this graph by generating the graphs of $f(t) = \sin 10t$ and of $g(t) = \cos 20t$ separately and then tracking the values of $x = \sin 10t$ and $y = \cos 20t$ as $t$ varies. This same technique can be used to conceptualize graphs of phase space, $(x,y) = (f(u), f'(u))$, in differential equations.

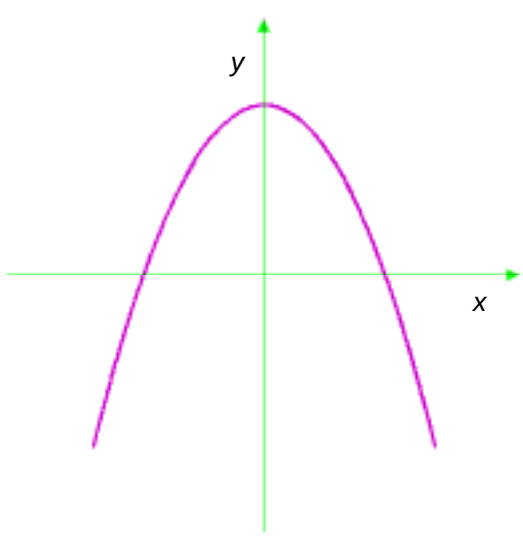



*Figure 7*. Graph of $(x,y) = (\sin 10t, \cos 20t)$, $0 \le t \le 1$.

Covariation can also support thinking about curves in space. To continue the previous example, imagine that $t$ in Figure 7 is actually an axis, coming straight at your eyes. If you now keep track of $t$ as well as $x$ and $y$, you get a sense that each point on the graph in Figure 7 is actually some distance toward you from the page. If you rotate your position relative to the graph so that you can see an axis that is perpendicular to $x$-$y$, then you have engendered an image like that in Figure 8.

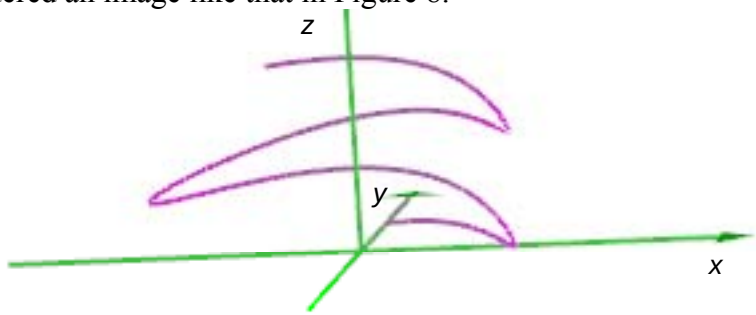



*Figure 8***.** Graph of $(x,y,z) = (\sin 10t, \cos 20t, t)$, $0 \le t \le 1$. As $t$ varies, points in Figure 7 with coordinates $(\sin 10t, \cos 20t)$ are projected $t$ units perpendicularly from the $x$-$y$ plane.

Finally, ideas of covariation can help students visualize functions of more than one variable. For example, we can envision the behavior of $z = f(x,y)$ in a multitude of ways, such as thinking of $y$ (or $x$) as a parameter. The graph of $f$, then, can be visualized as being generated by a family of functions $z = f_y(x)$ as $y$ varies. Figure 9 shows three successive graphs corresponding to $z=x^3+yx$ at $y = {}^-2$, at $y = {}^-1$, and at $y = 1$, where in each graph, $x$ varies from ${}^-2$ to 2. Figure 10 shows the surface swept out by $f_y(x) = x^3 + yx$ as $y$ varies continuously from -2 to 2, thus generating the graph of $f(x,y) = x^3 + yx$, ${}^-2 \le x \le 2$, ${}^-2 \le y \le 2$.

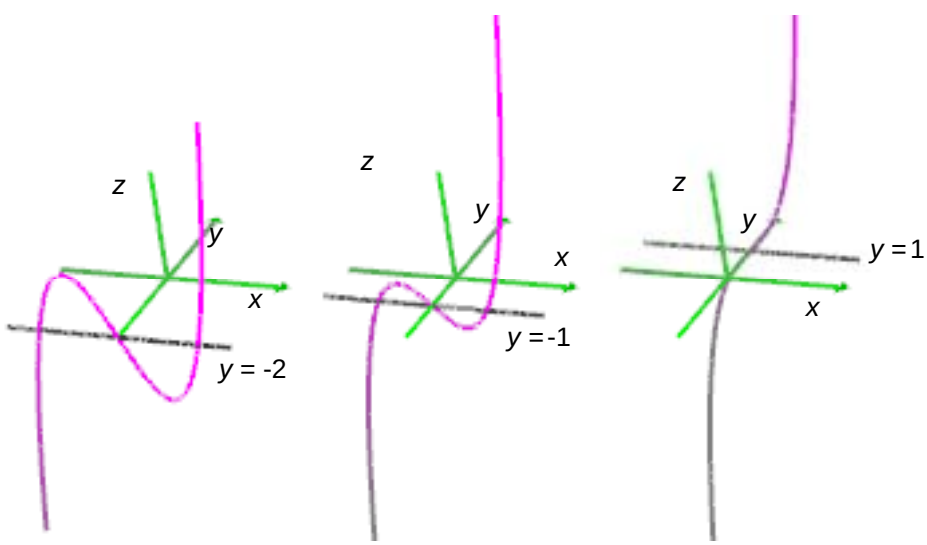
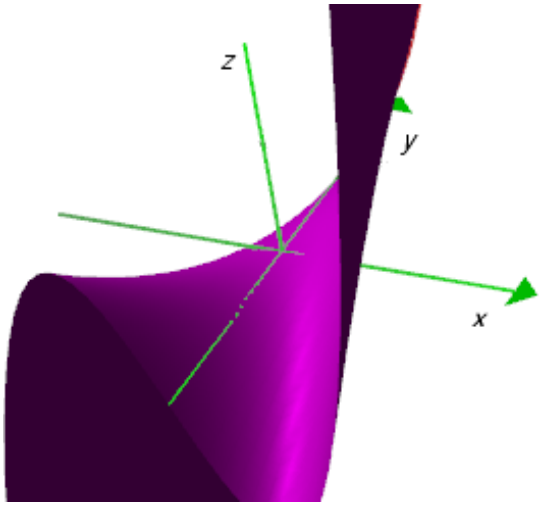



*Figure 9.* Graphs of $z = f_y(x)$ for $y = {}^-2$, $y = {}^-1$, and $y = 1$.




*(b)*

*Figure 10.* Graphs of $z = f(x,y)$, $^{-}2 \leq x \leq 2$, $^{-}2 \leq y \leq 2$.

## Concluding Remarks

A mature function understanding that is revealed by students' using functions fluidly, flexibly, and powerfully is typically associated with strong conceptual underpinnings. Promoting this conceptual structure in students' understanding may be achieved through both curriculum and instruction including tasks, prompts, and projects that promote and assess the development of these "ways of thinking" in students. We advocate for greater emphasis on enculturating students into using the language of function in order to develop facility in speaking about functions as entities that accept input and produce output, a more conceptual orientation to teaching function inverse and composition, the inclusion of tasks requiring simultaneous judgments about entire intervals of input or output values, and the development of students' ability to mentally run through a continuum of input values while imagining the changes in the output values, with explicit efforts to also promote, at the developmentally appropriate time, the covariational reasoning abilities described in this chapter. Our work also suggests that students would benefit from explicit efforts to promote their understanding of function notation. Additionally, we call for evaluations of students' mathematical development and readiness to include assessments that measure the foundational reasoning abilities needed for a robust function conception. As one example, when teaching calculus I, we begin the semester by assessing students' function understanding. This provides useful knowledge for our selecting and creating tasks to address their misconceptions and promote the reasoning abilities and understandings that we have described in this chapter. We have found that the time spent at the beginning of calculus to strengthen students' function conceptions is crucial for their understanding the major ideas of calculus (For further reading on how the covariation perspective to teaching functions influences students understanding of ideas of calculus, see Thompson and Silverman, this volume). (Note that the precalculus concept assessment instrument (PCA) and calculus modules mentioned in this chapter can be acquired by contacting the second author at marilyn.carlson@asu.edu.) You may also find it useful to assess your students' thinking

and reasoning on tasks that we have discussed in this chapter. Lastly, we advocate that you regularly pose questions and engage your students in tasks that will allow you to gauge your students' development in understanding major ideas of your courses. This instructional perspective will require that you have clarity on the mathematical thinking, understandings, and problem solving behaviors that your students need to acquire to advance their mathematical development. It also sets up a challenge for you to scaffold your instruction based on what your students know and understand, but should in turn lead to greater success for your students and a more rewarding instructional experience for you.

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